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The Vector QD Algorithm for Smooth Functions (f, f')

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We deal with the function $z \mapsto (f(z), f'(z))$ where $f(z)=\sum_{i \geqslant 0} a_i z^i$, $(a_i \in \mathbb{C})$ with $\lim_{i \to \infty} a_{i+1} \times a_{i-1}/(a_i)^2 = q$. We investigate the convergence of the vector QD algorithm. We give the asymptotic behaviour of the generalized Hankel determinants. A convergence result on the vector orthogonal polynomials is proved. © 1996 Academic Press, Inc.

1. INTRODUCTION

Let us consider a function f analytic at 0. Let us give an arbitrary polynomial v of degree q and an integer p. It is well known that there is a rational function with a numerator of degree p and $z^q v(z^{-1})$ as denominator whose series expansion coincides with that of f up to degree p. Such a rational function is called a Padé type approximant of f ([2]). For improving the degree of approximation, the polynomial v must be well chosen. This study leads to the polynomials $(P_r^{(s)})_{r,s}$. These polynomials are defined by (2). The polynomial $P_r^{(s)}$ exists and is unique if and only if the Hankel determinant

$$H_r^{(s)} = \begin{vmatrix} a_s & \cdots & a_{r+s-1} \\ \vdots & & \vdots \\ a_{s+r-1} & \cdots & a_{s+2r-2} \end{vmatrix} \tag{1}$$

does not vanish. Under this condition, $P_r^{(s)}$ has the following expression

$$P_r^{(s)} = \frac{H_r^{(s)}(x)}{H_r^{(s)}} = \frac{\begin{vmatrix} a_s & \cdots & a_{r+s-1} & 1 \\ \vdots & & \vdots & \vdots \\ a_{s+r-1} & \cdots & a_{s+2r-2} & x^{r-1} \\ a_{s+r} & \cdots & a_{s+2r-1} & x^r \end{vmatrix}}{\begin{vmatrix} a_s & \cdots & a_{r+s-1} \\ \vdots & & \vdots \\ a_{s+r-1} & \cdots & a_{s+2r-2} \end{vmatrix}}. \tag{2}$$

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If $H_r^{(s)} \neq 0$, the polynomial $\widetilde{P}_r^{(s)}$, i.e. $\widetilde{P}_r^{(s)}(x) = x^r P_r^{(s)}(1/x)$, is the denominator of the Padé approximant $[r+s-1/r]_f$ of f. Moreover, s being fixed, the family $(P_r^{(s)})_{r \geqslant 0}$ is the family of formal orthogonal polynomials associated to the functional $a^{(s)}$

$$a^{(s)}(x^i) = a_{i+s}. \tag{3}$$

On the other hand, we have the following recurrence relation

$$P_{r+1}^{(s)}(x) = xP_r^{(s+1)}(x) - q_{r+1}^{(s)} P_r^{(s)}(x)\ r \geqslant 0,\ s \geqslant 0,\ P_0^{(s)} \equiv 0\ (s \geqslant 0) \tag{4}$$

with

$$q_r^{(s)} = \frac{H_{r+1}^{(s+1)} \times H_r^{(s)}}{H_{r+1}^{(s)} \times H_r^{(s+1)}}. \tag{5}$$

The numbers $(q_r^{(s)})$ are computed by using the quotient difference algorithm (QD algorithm) of Rutishauer ([5], p. 609).

Lots of convergence results on the polynomials $P_r^{(s)}$ exist. A review of results can be found in [3]. It can be remarked that if the behaviour of the sequence $(q_{r+1}^{(s)})_{s \geqslant 0}$ (r is fixed) is known, by using (4) and arguing by recurrence on r, convergence results for the sequences $(P_r^{(s)})_{s \geqslant 0}$ can be proved. This idea occurs in [6]. The interesting fact is that the relation (4) is still valid in the vector case. The Hankel determinants are replaced by generalized Hankel determinants:

$$\text{if } a_i \in \mathbb{C}^d \text{ and } r = nd + k\ (0 \leqslant k \leqslant d)$$

$$H_r^{(s)} = \begin{vmatrix} a_s & \cdots & a_{r+s-1} \\ \vdots & & \vdots \\ a_{s+n-1} & \cdots & a_{s+n+r-2} \\ a_{s+n}^{(k)} & \cdots & a_{s+n+r-1}^{(k)} \end{vmatrix} \tag{6}$$

(each vector row represents the d scalar rows of the components, and the last row the k first components). The polynomial $\widetilde{P}_r^{(s)}(x)$ is the denominator (see Section 3.1) of a vector rational function which approaches the series $\sum_{n \geqslant 0} a_n z^n$ in the Padé sense. In Section 3, the definition of a vector Padé approximant will be recalled (for more details and for comparison with other simultaneous rational approximations see [9]). The quantities $q_r^{(s)}$ can be computed by using a vector quotient-difference algorithm ([9]).

We are going to exploit this idea in the case of a function $z \mapsto (f(z), f'(z))$, $f: \mathbb{C} \to \mathbb{C}$. Why do we consider this choice? There are two motivations:

• the problem to approximate a function and its first derivative (this problem was already considered in [1, 4]);

• if we consider the pair (f, f') we have a link between the coefficients of the two series.

This paper is organized as follows. In Section 2 we come back to the scalar case. In Section 3 we deal with the function $z \mapsto (f(z), f'(z))$ where $f(z) = \sum_{i \geqslant 0} a_i z^i$ with the assumption $\lim_{i \to \infty} a_{i+1} \times a_{i-1}/(a_i)^2 = q$. A result on the asymptotic behaviour of the generalized Hankel determinants is proved. Finally, we investigate the convergence of the vector orthogonal polynomials.

2. THE SCALAR CASE

2.1. *Meromorphic Functions*

Let us quote the following result:

THEOREM 2.1 [5, p. 626]. *Let the function f be analytic at $z=0$ and meromorphic in the disk D: $|z| < \sigma$ and let its poles $z_i = u_i^{-1}$ in D, which may be finite or infinite in number, be numbered such that*

$$0 < |z_1| \leqslant |z_2| \leqslant \cdots < \sigma,$$

each pole occurring as many times in the sequence (z_k) as indicated by its order. Let the Taylor series of f at 0 be normal (i.e. $H_r^{(s)} \neq 0$ for all $r \geqslant 0$ and $s \geqslant 0$). Then for every r such that $|z_r| < |z_{r+1}|$ the polynomials $P_r^{(s)}$ associated with the Taylor series satisfy

$$\lim_{s \to \infty} P_r^{(s)}(x) = (x - u_1)(x - u_2) \cdots (x - u_r),$$

uniformly in x in every bounded set.

Remark 2.1. By using (4), we can obtain a similar result. However, the assumptions must be stronger. As a matter of fact, still according to [5], if $|z_{r-1}| < |z_r| < |z_{r+1}|$, then we have $\lim_{s \to \infty} q_r^{(s)} = u_r$.

Starting from $P_0^{(s)} \equiv 1$ $(s \geqslant 0)$, by virtue of (4), we get

$$P_1^{(s)} = x P_0^{(s+1)}(x) - q_1^{(s)} P_0^{(s)}(x) = x - q_1^{(s)} \tag{7}$$

and thus $\lim_{s \to \infty} P_1^{(s)}(x) = x - u_1$ uniformly on the compact subsets of $\mathbb{C}$. Assuming that

$$\lim_{s \to \infty} P_r^{(s)}(x) = (x - u_1)(x - u_2) \cdots (x - u_r)$$

uniformly on the compact subsets of $\mathbb{C}$, and writing

$$P_{r+1}^{(s)}(x) = xP_r^{(s+1)}(x) - q_{r+1}^{(s)} P_r^{(s)}(x),$$

we obtain

$$\lim_{s\to\infty} P_{r+1}^{(s)}(x) = x(x-u_1)\cdots(x-u_r) - u_{r+1}(x-u_1)\cdots(x-u_r) \tag{8}$$

$$= (x-u_1)\cdots(x-u_r)(x-u_{r+1}) \tag{9}$$

uniformly on compact subsets of $\mathbb{C}$.

2.2. *Functions with Smooth Maclaurin Series Coefficients*

Let us consider the formal power series $f(z) = \sum_{j\geqslant 0} a_j z^j$ with $a_j \in \mathbb{C}\setminus\{0\}$ and

$$\lim_{i\to\infty} \frac{a_{i+1}\times a_{i-1}}{(a_i)^2} = q \in \mathbb{C}.$$

DEFINITION 2.1. Such a function is called a function with smooth Maclaurin series coefficients.

If $|q| < 1$, f is an entire function of zero order.

Lubinsky obtained the following result on the Toeplitz determinants $D(s/r)$ (i.e. $H_r^{(s)} = (-1)^{r(r-1)/2} D(s+r-1/r)$):

THEOREM 2.2 [6]. *Let us assume that* $f(z) = \sum_{j\geqslant 0} a_j z^j$ *with* $a_j \neq 0$ *for* j *large enough and*

$$\lim_{i\to\infty} \frac{a_{i+1}\times a_{i-1}}{(a_i)^2} = q \in \mathbb{C}.$$

Then we have

$$\lim_{s\to\infty} \frac{D(s/r)}{(a_s)^r} = \prod_{j=1}^{r-1} (1-q^j)^{r-j} \ (r \textit{ is fixed}).$$

Let us note that if q is not a root of unity, we are allowed to do ratios of Toeplitz determinants. Thus

COROLLARY 2.1. *With the previous assumptions and if q is not a root of unity, we have*

$$q_r^{(s)} = \frac{a_{r+s}}{a_{r+s-1}}(1+o(1)),$$

r being fixed and $s \to \infty$.

This result occurs in [6]. We have

$$q_r^{(s)} = \frac{H_r^{(s+1)} \times H_{r-1}^{(s)}}{H_r^{(s)} \times H_{r-1}^{(s+1)}} \tag{10}$$

$$= \frac{D(r+s/r) \times D(s+r-2/r-1)}{D(r+s-1/r) \times D(r+s-1/r-1)} \tag{11}$$

$$= \frac{\dfrac{D(r+s/r)}{(a_{r+s})^r}}{\dfrac{D(r+s+1)}{(a_{r+s-1})^r}} \times \frac{\dfrac{D(r+s-2/r-1)}{(a_{r+s-2})^{r-1}}}{\dfrac{D(r+s-1/r-1)}{(a_{r+s-1})^{r-1}}}$$

$$\times \left(\frac{a_{r+s} \times a_{r+s-2}}{(a_{r+s-1})^2}\right)^{r-1} \times \frac{a_{r+s}}{a_{r+s-1}} \tag{12}$$

Applying Theorem 2.2, we obtain the asymptotic behaviour of $q_r^{(s)}$.

Remark 2.2. We can prove directly the result of the corollary by using the rules of the QD algorithm ([5], p. 609). Let us consider, with the usual notation, the property

$$(R_m)\begin{cases} q_m^{(n)} \overset{n\to\infty}{=} \dfrac{a_{m+n}}{a_{m+n-1}} q^{m-1}(1+o(1)) \\ e_m^{(n)} \overset{n\to\infty}{=} \dfrac{a_{m+n}}{a_{m+n-1}} (q^m-1)(1+o(1)) \end{cases} \tag{13}$$

($q \in \mathbb{C}^*$ and is not a root of unity). The initialization of the QD algorithm gives

$$e_1^{(n)} = q_1^{(n+1)} - q_1^{(n)} = \frac{a_{n+2}}{a_{n+1}} - \frac{a_{n+1}}{a_n}. \tag{14}$$

Thus

$$e_1^{(n)} = \frac{a_{n+1}}{a_n}\left(\frac{a_{n+2} \times a_n}{(a_{n+1})^2} - 1\right) \overset{n\to\infty}{=} \frac{a_{n+1}}{a_n}(q-1)(1+o(1)), \tag{15}$$

(R_1) is true. Let us now assume that (R_m) is true. Then

$$q_{m+1}^{(n)} = \frac{e_m^{(n+1)}}{e_m^{(n)}} q_m^{(n+1)} \tag{16}$$

$$\overset{n\to\infty}{=} \frac{\frac{a_{m+n+1}}{a_{m+n}}(q^m-1)(1+o(1))}{\frac{a_{m+n}}{a_{m+n-1}}(q^m-1)(1+o(1))} \times \frac{a_{m+n+1}}{a_{m+n}} q^{m-1}(1+o(1)) \tag{17}$$

$$\overset{n\to\infty}{=} \frac{a_{m+n+1}\times a_{m+n-1}}{(a_{m+n})^2} \times \frac{a_{m+n+1}}{a_{m+n}} q^{m-1}(1+o(1)) \tag{18}$$

$$\overset{n\to\infty}{=} \frac{a_{m+n+1}}{a_{m+n}} q^m(1+o(1)) \tag{19}$$

and

$$e_{m+1}^{(n)} = [q_{m+1}^{(n+1)} - q_{m+1}^{(n)}] + e_m^{(n+1)} \tag{20}$$

$$e_{m+1}^{(n)} \overset{n\to\infty}{=} \left[\frac{a_{m+n+2}}{a_{m+n+1}} q^m(1+o(1)) - \frac{a_{m+n+1}}{a_{m+n}} q^m(1+o(1))\right] + \frac{a_{m+n+1}}{a_{m+n}}(q^m-1)(1+o(1))$$

$$e_{m+1}^{(n)} \overset{n\to\infty}{=} \frac{a_{m+n+1}}{a_{m+n}}\left[\frac{a_{m+n+2}\times a_{m+n}}{(a_{m+n+1})^2} q^m(1+o(1)) - q^m(1+o(1)) + (q^m-1)(1+o(1))\right]$$

$$e_{m+1}^{(n)} \overset{n\to\infty}{=} \frac{a_{m+n+1}}{a_{m+n}}[q^{m+1}(1+o(1)) - q^m(1+o(1)) + (q^m-1)(1+o(1))],$$

and finally

$$e_{m+1}^{(n)} \overset{n\to\infty}{=} \frac{a_{m+n+1}}{a_{m+n}}(q^{m+1}-1)(1+o(1)). \tag{21}$$

(R_{m+1}) is true. In conclusion, (R_m) is true for all $m \in \mathbb{N}^*$.

Let us come back to the recurrence relation (4). It can be written as

$$\tilde{P}_{r+1}^{(s)}(x) = \tilde{P}_r^{s+1}(x) - xq_{r+1}^{(s)}\tilde{P}_r^{(s)} \tag{22}$$

and thus

$$\tilde{P}_{r+1}^{(s)}\left(x\frac{a_{r+s}}{a_{r+s+1}}\right) = \tilde{P}_r^{(s+1)}\left(x\frac{a_{r+s}}{a_{r+s+1}}\right) - xq_{r+1}^{(s)}\frac{a_{r+s}}{a_{r+s+1}}\tilde{P}_r^{(s)}\left(x\frac{a_{r+s-1}}{a_{r+s}}\frac{(a_{r+s})^2}{a_{r+s-1}\times a_{r+s+1}}\right).$$

Lubinsky proved then, by recurrence and using the asymptotic expansion of $q_{r+1}^{(s)}$ as $s\to\infty$, that

$$\lim_{s\to\infty}\tilde{P}_r^{(s)}\left(x\frac{a_{r+s-1}}{a_{r+s}}\right) = B_r(x) \tag{23}$$

uniformly on compact subsets of $\mathbb{C}$, where B_r denotes the rth Rogers-Szegö polynomial ([6, 7]). These polynomials are defined by

$$B_{r+1}(x) = B_r(x) - xq^r B_r(xq^{-1})\ (B_0 \equiv 1). \tag{24}$$

Let us now deal with the vector case.

3. THE VECTOR CASE

3.1. *Notation* [9]

Let us consider two analytic functions at zero, $f_1(z) = \sum_{i\geqslant 0} a_i z^i$ and $f_2(z) = \sum_{i\geqslant 0} b_i z^i$. We put $F(z) = (f_1(z), f_2(z))$.

Definition 3.1. The generalized Hankel determinants associated to F are

$$H_r^{(s)} = \begin{vmatrix} a_s & \cdots & a_{r+s-1} \\ b_s & \cdots & b_{r+s-1} \\ a_{s+1} & \cdots & a_{r+s} \\ b_{s+1} & \cdots & b_{r+s} \\ \vdots & \vdots & \vdots \\ a_{s+n-1} & \cdots & a_{r+s+n-2} \\ b_{s+n-1} & \cdots & b_{r+s+n-2} \end{vmatrix} \quad \text{if} \quad r = 2n \tag{25}$$

and

$$H_r^{(s)} = \begin{vmatrix} a_s & \cdots & a_{r+s-1} \\ b_s & \cdots & b_{r+s-1} \\ a_{s+1} & \cdots & a_{r+s} \\ b_{s+1} & \cdots & b_{r+s} \\ \vdots & \vdots & \vdots \\ a_{s+n-1} & \cdots & a_{r+s+n-2} \\ b_{s+n-1} & \cdots & b_{r+s+n-2} \\ a_{s+n} & \cdots & a_{r+s+n-1} \end{vmatrix} \quad \text{if} \quad r = 2n+1 \tag{26}$$

Given any $(p, q) \in \mathbb{N}^2$, we seek a polynomial Q of degree q and two polynomials P_1 and P_2 of degree p, such that

$$F(z) - \frac{\mathscr{P}}{Q}(z) = 0(z^{p+[q/2]+1}) \qquad \text{as} \quad z \to 0 \tag{27}$$

($\mathscr{P} = (P_1, P_2)$, $[q/2]$ is the integer part of $q/2$).

DEFINITION 3.2. The vector $\mathscr{P}/Q = (P_1/Q, P_2/Q)$ of rational functions is called the vector Padé approximant of degree (p, q) to the vector of formal power series $F = (f_1, f_2)$, and is denoted by $[p/q]_F$.

For example, if $r = 2n$ and $H_{2n}^{(s)}$ does not vanish, the denominator of the vector Padé approximant $[r+s-1/r]_F$ is the polynomial $\tilde{P}_r^{(s)}$ where:

$$P_r^{(s)} = \frac{\begin{vmatrix} a_s & \cdots & a_{r+s} \\ b_s & \cdots & b_{r+s} \\ a_{s+1} & \cdots & a_{r+s+1} \\ b_{s+1} & \cdots & b_{r+s+1} \\ \vdots & \vdots & \vdots \\ a_{s+n-1} & \cdots & a_{s+r+n-1} \\ b_{s+n-1} & \cdots & b_{s+r+n-1} \\ 1 & \cdots & x^r \end{vmatrix}}{\begin{vmatrix} a_s & \cdots & a_{r+s-1} \\ b_s & \cdots & b_{r+s-1} \\ a_{s+1} & \cdots & a_{r+s} \\ b_{s+1} & \cdots & b_{r+s} \\ \vdots & \vdots & \vdots \\ a_{s+n-1} & \cdots & a_{r+s+n-2} \\ b_{s+n-1} & \cdots & b_{r+s+n-2} \end{vmatrix}}. \tag{28}$$

We have

$$F(z) - [r+s-1/r]_F(z) = O(z^{r+s+n}) \qquad \text{as} \quad z \to 0,\ r = 2n+k\ (k=0 \text{ or } 1). \tag{29}$$

Let us now deal with the function $F(z) = (f(z), f'(z))$ where $f(z) = \sum_{i \geq 0} a_i z^i$ $(a_i \in \mathbb{C}^*)$. Let q_j be the quantity $a_{j-1} \times a_{j+1}/(a_j)^2$. We assume that

$$\lim_{j \to \infty} q_j = q \in \mathbb{C}. \tag{30}$$

First of all, we shall establish a result on the asymptotic behaviour of the determinant $H_r^{(s)}$ when $s \to \infty$.

3.2. *Asymptotic Behaviour of* $H_r^{(s)}$

In the scalar case the behaviour of the Toeplitz determinants was compared to the product of the elements of the diagonal. If we consider Hankel determinants, we have to take the product of the elements of the antidiagonal. We now state our result:

THEOREM 3.1. *Let* $f(z) = \sum_{i \geq 0} a_i z^i$ $(a_i \in \mathbb{C}^*)$ *be a formal power series. Let us assume that* $\lim_{j \to \infty} q_j = q \in \mathbb{C}$, *where* $q_j = a_{j-1} \times a_{j+1}/(a_j)^2$. *Let r be an integer,* $r = 2n+k$ $(k=0$ *or* $1)$. *The two following quantities have a limit, denoted* $\pi_r(q)$, *as* $s \to \infty$ *and r is fixed*

$$\frac{H_r^{(s)}}{s(a_{s+n})^2 \cdots (a_{s+r-1})^2} \qquad (r \text{ is even})$$

$$\frac{H_r^{(s)}}{a_{s+n}(a_{s+n+1})^2 \cdots (a_{s+r-1})^2} \qquad (r \text{ is odd}).$$

Proof. With the previous notation, we have here $b_s = (s+1)\, a_{s+1}$. Thus, if r is even, by doing linear combinations between the rows, we obtain

$$H_r^{(s)} = \begin{vmatrix} a_s & a_{s+1} & \cdots & a_{r+s-1} \\ a_{s+1} & 2a_{s+2} & \cdots & ra_{r+s} \\ a_{s+1} & a_{s+2} & \cdots & a_{r+s} \\ \vdots & \vdots & & \vdots \\ (n-1)\, a_{s+n-1} & na_{s+n} & \cdots & (r+n-2)\, a_{r+s+n-2} \\ a_{s+n-1} & a_{s+n} & \cdots & a_{r+s+n-2} \\ (s+n)\, a_{s+n} & (s+n+1)\, a_{s+n+1} & \cdots & (r+s+n-1)\, a_{s+n-1+r} \end{vmatrix}$$

It can be remarked that s appears as a factor only for the coefficients of the last row. In the same way, if r is odd, there is no coefficient with s as factor. If $i=2\theta_i+k_i$ ($k_i=0$ or 1), the coefficient of the ith row and jth column of $H_r^{(s)}$ is

$$a_{s+\theta_i+j-1} \qquad \text{if} \quad k_i=1 \tag{31}$$

$$(\theta_i+j-1)\, a_{s+\theta_i+j-1} \qquad \text{if} \quad k_i=0 \qquad \text{and} \qquad i\neq 2n \tag{32}$$

$$(s+\theta_i+j-1)\, a_{s+\theta_i+j-1} \qquad \text{if} \quad i=2n. \tag{33}$$

Let us multiply the ith row by $(a_{r+s}/a_{r+s-1})^{r+1-i}$ and the jth column by $(a_{r+s}/a_{r+s-1})^{-j}$, for $i=1, ..., r$, $j=1, ..., r$. These multiplications do not change the value of the determinant.

Let us put

$$\mathscr{D}_1=\frac{H_r^{(s)}}{s(a_{s+n})^2\cdots(a_{s+r-1})^2} \qquad (r \text{ is even}), \tag{34}$$

and

$$\mathscr{D}_2=\frac{H_r^{(s)}}{a_{s+n}(a_{s+n+1})^2\cdots(a_{s+r-1})^2} \qquad (r \text{ is odd}). \tag{35}$$

We can see that the coefficient of the ith row and jth column is in $\mathscr{D}_1$ or $\mathscr{D}_2$

$$\left(\frac{a_{s+\theta_i+j-1}}{a_{s+\theta_i+r-i}}\right)\left(\frac{a_{r+s}}{a_{r+s+1}}\right)^{r+1-(i+j)} \tag{36}$$

with a multiplying factor which does not depend on s, except for the last row of $\mathscr{D}_1$ where this factor is of the form $(s+h)/s$, which tends to 1 when $s\to\infty$. Putting $t=(i+j)-(r+1)$ and $m=s+\theta_i+r-i$, we have

$$\frac{a_{s+\theta_i+j-1}}{a_{s+\theta_i+r-i}}=\frac{a_{m_i+t}}{a_{m_i}}. \tag{37}$$

Let us assume for example that $t>0$. Then

$$\frac{a_{m_i+t}}{a_{m_i}}=\prod_{l=0}^{t-1}\frac{a_{m_i+l+1}}{a_{m_i+l}} \tag{38}$$

but

$$\frac{a_{m_i+l+1}}{a_{m_i+l}}=q_{m_i+l}\cdots q_{m_i+1}\frac{a_{m_i+1}}{a_{m_i}} \tag{39}$$

thus

$$\frac{a_{m_i+t}}{a_{m_i}} \overset{s\to\infty}{=} q^{t(t-1)/2} \left(\frac{a_{m_i+t}}{a_{m_i}}\right)^t (1+o(1)). \tag{40}$$

We now have to examine

$$\left(\frac{a_{m_i+1}}{a_{m_i}} \times \frac{a_{r+s-1}}{a_{r+s}}\right)^t. \tag{41}$$

Starting from $a_{r+s}/a_{r+s-1} = a_{m_i+l+1}/a_{m_i+l}$ with $l = i - \theta_i - 1$, we obtain

$$\frac{a_{r+s}}{a_{r+s-1}} = \left(\prod_{k=1}^{i-\theta_i-1} q_{m_i+k}\right) \frac{a_{m_i+1}}{a_{m_i}} \tag{42}$$

and

$$\left(\frac{a_{m_i+1}}{a_{m_i}} \times \frac{a_{r+s-1}}{a_{r+s}}\right)^t = \left(\prod_{k=1}^{i-\theta_i-1} (q_{m_i+k})^{-1}\right)^t \tag{43}$$

$$\overset{s\to\infty}{=} q^{(\theta_i+1-i)\,t}(1+o(1)). \tag{44}$$

Finally, by taking the limit in $\mathscr{D}_1$ or in $\mathscr{D}_2$, we obtain a determinant whose coefficient of the ith row and the jth column is, with a multiplying factor,

$$q^{(\theta_i+1-i)\,t} \times q^{t(t-1)/2} \tag{45}$$

$$t = i + j - (r+1) \qquad \text{and} \qquad i = 2\theta_i + k_i. \quad \blacksquare$$

Remark 3.1. 1. $\pi_r(q)$ is a rational function of q;

2. $\pi_r(q)$ depends only on q. More precisely, $\pi_r(q)$ can be computed starting from the series $f(z) = \sum_{i\geqslant 0} q^{i(i-1)/2} z^i$;

3. for example, it can be seen that

(a) $\pi_2(q) = q - 1$;

(b) $\pi_3(q) = -(q-1)^2$;

(c) $\pi_4(q) = (q-1)^5 (q+1)$;

(d) $\pi_5(q) = (q-1)^8 (q+1)^2$;

(e) $\pi_6(q) = (q^3-1)(q+1)^4 (q-1)^{12}$;

(f) $\pi_7(q) = -(q^3-1)^2 (q+1)^6 (q-1)^{16}$.

3.3. *Asymptotic Expansion for the Vector QD Algorithm*

Let us now establish the asymptotic of the vector QD algorithm.

COROLLARY 3.1. *Let us assume that* $\pi_r(q) \neq 0$ *and* $\pi_{r-1}(q) \neq 0$ $(r = 2n + k,\ k = 0$ *or* $1)$. *Then, we have*

$$q_r^{(s)} \overset{s\to\infty}{=} q^{r-1} \frac{a_{s+n+1}}{a_{s+n}} (1 + o(1)).$$

Proof. Let us prove it when r is even. We have

$$q_r^{(s)} = \frac{\dfrac{H_r^{(s+1)}}{(s+1)(a_{s+n+1})^2 \cdots (a_{s+r})^2} \times \dfrac{H_{r-1}^{(s)}}{a_{s+n-1}(a_{s+n})^2 \cdots (a_{s+r-2})^2}}{\dfrac{H_r^{(s)}}{s(a_{s+n})^2 \cdots (a_{s+r-1})^2} \times \dfrac{H_{r-1}^{(s+1)}}{a_{s+n}(a_{s+n+1})^2 \cdots (a_{s+r-1})^2}}$$
$$\times \frac{(s+1)(a_{s+n+1})^2 \cdots (a_{s+r})^2 \times a_{s+n-1}(a_{s+n})^2 \cdots (a_{s+r-2})^2}{s(a_{s+n})^2 \cdots (a_{s+r-1})^2 \times a_{s+n}(a_{s+n+1})^2 \cdots (a_{s+r-1})^2}.$$

and by virtue of Theorem 3.1

$$q_r^{(s)} \overset{s\to\infty}{=} (1 + o(1)) \left(\frac{a_{s+r}}{a_{s+r-1}}\right)^2 \times \frac{a_{s+n-1}}{a_{s+n}}. \tag{46}$$

Moreover

$$\left(\frac{a_{s+r}}{a_{s+r-1}}\right)^2 \times \frac{a_{s+n-1}}{a_{s+n}} = \left(\frac{a_{s+r}}{a_{s+n}}\right)^2 \times \left(\frac{a_{s+n-1}}{a_{s+r-1}}\right)^2 \times \frac{a_{s+n}}{a_{s+n-1}}, \tag{47}$$

$$\frac{a_{s+r}}{a_{s+n}} = \prod_{l=0}^{n-1} \left(q_{s+n+l} \times q_{s+n+l-1} \times \cdots \times q_{s+n+1} \times \frac{a_{s+n+1}}{a_{s+n}} \right) \tag{48}$$

$$\overset{s\to\infty}{=} \left(\frac{a_{s+n+1}}{a_{s+n}}\right)^n q^{n(n-1)/2}(1 + o(1)), \tag{49}$$

and

$$\frac{a_{s+r-1}}{a_{s+n-1}} = \frac{a_{s+n-1+n}}{a_{s+n-1}} \tag{50}$$

$$\overset{s\to\infty}{=} \left(\frac{a_{s+n}}{a_{s+n-1}}\right)^n q^{n(n-1)/2}(1 + o(1)). \tag{51}$$

Thus

$$\left(\frac{a_{s+r}}{a_{s+r-1}}\right)^2 \times \frac{a_{s+n-1}}{a_{s+n}} \overset{s\to\infty}{=} \left(\frac{a_{s+n+1}\times a_{s+n-1}}{(a_{s+n})^2}\right)^{2n} \times \frac{a_{s+n}}{a_{s+n-1}}(1+o(1))$$

$$\overset{s\to\infty}{=} \left(\frac{a_{s+n+1}\times a_{s+n-1}}{(a_{s+n})^2}\right)^{2n-1} \times \frac{a_{s+n+1}}{a_{s+n}}(1+o(1))$$

$$\overset{s\to\infty}{=} q^{r-1}\frac{a_{s+n+1}}{a_{s+n}}(1+o(1)). \quad \blacksquare$$

3.4. *Convergence of the Vector Orthogonal Polynomials*

The following relation holds

$$\tilde{P}_{r+1}^{(s)}(x) = \tilde{P}_r^{(s+1)}(x) - xq_{r+1}^{(s)}\tilde{P}_r^{(s)}(x). \tag{52}$$

Thus, using the asymptotic of the vector QD algorithm, it is not difficult to obtain asymptotics for the vector orthogonal polynomials. But, the main difference with the scalar case, is that we have to distinguish between the cases r even and r odd. Let us now state our last result.

COROLLARY 3.2. *Assume that $\pi_r(q) \neq 0$ for all $r \geqslant 0$. Define the polynomials V_n by*

$$V_{n+1}(x) = x^2q^{4n}V_n(xq^{-1}) - xq^{2n}(q+1)\, V_n(x) + V_n(qx)\ (V_0 \equiv 1).$$

Then

$$\lim_{s\to\infty} \tilde{P}_{2n}^{(s)}\left(x\frac{a_{s+n}}{a_{s+n+1}}\right) = V_n(x)$$

uniformly on compact subsets of $\mathbb{C}$.

Proof. Putting $r = 2n$ and using Corollary 3.1, we have

$$\tilde{P}_{r+1}^{(s)} \overset{s\to\infty}{=} \tilde{P}_r^{(s+1)}\left(x\frac{a_{s+n+1}}{a_{s+n+2}} \times \frac{a_{s+n+2}\times a_{s+n}}{(a_{s+n+1})^2}\right)$$

$$- xq^r(1+o(1))\,\tilde{P}_r^{(s)}\left(x\frac{a_{s+n}}{a_{s+n+1}}\right).$$

Then if

$$\lim_{s\to\infty} \tilde{P}_r^{(s)}\left(x\frac{a_{s+n}}{a_{s+n+1}}\right) = P_1(x) \tag{53}$$

uniformly on compact subsets of $\mathbb{C}$, $\tilde{P}^{(s)}_{r+1}(x(a_{s+n}/a_{s+n+1}))$ converges uniformly on compact subsets of $\mathbb{C}$ to $P_2(x)$ such that

$$P_2(x) = P_1(xq) - xq^r P_1(x). \tag{54}$$

Moreover in the same way, $\tilde{P}^{(s)}_{r+2}(x(a_{s+n+1}/a_{s+n+2}))$ converges uniformly on compact subsets of $\mathbb{C}$ to $P_3(x)$ such that

$$P_3(x) = P_2(x) - xq^{r+1} P_2(xq^{-1}). \tag{55}$$

Finally

$$\begin{aligned} P_3(x) &= P_1(xq) - xq^r P_1(x) - xq^{r+1}(P_1(x) - xq^{r-1} P_1(xq^{-1})) \\ &= x^2 q^{2r} P_1(xq^{-1}) - xq^r(q+1)\, P_1(x) + P_1(qx). \quad \blacksquare \end{aligned}$$

Remark 3.2. We can note that $V_{n+1}(0) = V_n(0)$ for all $n \geqslant 0$. As $V_0 \equiv 1$, it follows that the zeros of V_n are nonzero.

Remark 3.3. Let us assume that $|q| < 1$. Then, f is an entire function of zero order. By using a theorem of Hurwitz (see [8], p. 119) and the fact that $\lim_{s \to \infty} \tilde{P}^{(s)}_{2n}(x(a_{s+n}/a_{s+n+1})) = V_n(x)$ uniformly on compact subsets of $\mathbb{C}$, it can be seen that the zeros of $\tilde{P}^{(s)}_{2n}$ (i.e. the poles of $[2n+s-1/2n]$) approach ∞ with rate a_{s+n}/a_{s+n+1} as $s \to \infty$. According to the integral formula of the rest $F(z) - [2n+s-1/2n]_F(z)$ ([8], p. 12), it follows that the sequence $([2n+s-1/2n]_F)_{s \geqslant 0}$ converges to F locally uniformly in $\mathbb{C}$.

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REFERENCES

1. G. Baker and M. P. Graves, Padé approximants. II. Extensions and applications, *in* "Encyclopedia of Mathematics," Addison–Wesley, Reading, MA, 1981.
2. C. Brezinski, "Padé-Type Approximants and General Orthogonal Polynomials," Birkhäuser-Verlag, Basel, 1980.
3. C. Brezinski and J. Van Iseghem, Padé approximations, *in* "Handbook of Numerical Analysis" (P. G. Ciarlet and J. L. Lions, Eds.), Vol. 3, Amsterdam, 1992, North-Holland.
4. P. J. Davis, "Interpolation and Approximation," Dover, New York, 1975.
5. P. Henrici, "Applied and Computational Complex Analysis," Wiley, New York, 1974.

6. D. S. Lubinsky, Uniform convergence of rows of the Padé table for functions with smooth Mac Laurin series coefficients, *Constr. Approx.* **3** (1987), 307–330.
7. G. Szegö, Ein Beitrag zur Theorie der Thetafunktionen: Sitzungsberichte der Preussichen Akademie der Wissenschaften, *Phys. Math.* **K1** (1926), 242–251.
8. E. C. Titchmarsh, "The Theory of Functions," 2nd ed., Oxford Univ. Press, Oxford, 1939.
9. J. Van Iseghem, "Approximants de Padé Vectoriels," thesis, University of Lille, 1987.